Pumping Lemma

A property of regular sets

- Weak version
- Strong version
- Applications

Pumping Lemma (weak)

If a language L is accepted by a DFA M with m states, then any string x in L with $|x| \geq m$ can be written as x = uvw such that

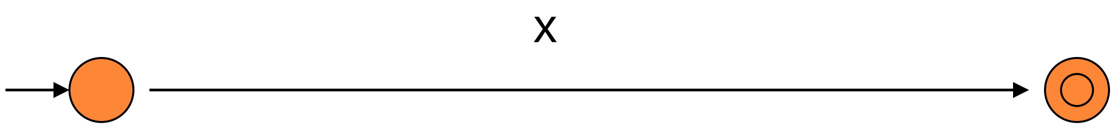
(1) $v \neq \varepsilon$, and

(2) uv^*w is a subset of L

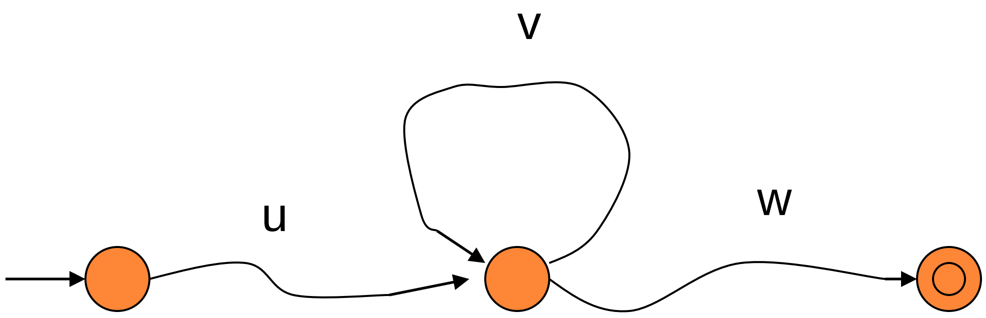
(i.e., for any $n \geq 0$, $uv^n w$ in L).

PROOF

- Consider the path associated with x (|x| > m).



Since |x| $\geq$ m, # of nodes on the path is
At least m+1. Therefore, there is a state
Appearing twice.



$v \neq \varepsilon$ because M is DFA

uw in L because there is a path associated with uw from initial state to a final state.

$uv^n w$ in L due to the same reason as above

$L=\{0^n \mid$ N IS A PRIME$\}$ IS NOT REGULAR.

Proof. For contradiction, suppose L is regular. So, $L=L(M)$ for some DFA M.

Let m be the number of states of M. Consider a prime $p > m$. By Pumping Lemma, $0 = uvw$ such that $v \neq \varepsilon$ and

uv^*w is a subset of L. Thus,

$p = |u| + |v| + |w|$

and for any $k \geq 0$, $|u|+k|v|+|w|$ is a prime.

For $k = 0$, $|u|+|w|$ is a prime.

For $k=|u|+|w|$, $|u|+k|v|+|w|$

$= (|u|+|w|)(1+|v|)$

is a prime. (-><-)

$L=\{0^i 1^i \mid i \geq 0\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider $0^m 1^m$.

By Pumping Lemma, $0^m 1^m$ = uvw such that $v \neq \varepsilon$ and for $n \geq 0$, $uv^n w$ in L.

Case 1. v is a substring of 0^m.

uw in L, but uw contains less 0's than 1's. (-><-)

Case 2. v is a substring of 1^m.

uw in L, but uw contains less 1's than 0's. (-><-)

Case 3. v contains both 0 and 1.

uvvw in L, but uvvw contains 10. (-><-)

PUMPING LEMMA (STRONG)

- If a language L is accepted by a DFA M with m states, then any string xyz in L with $|y| \geq m$ can be written as y = uvw such that (1) $v \neq \varepsilon$, and

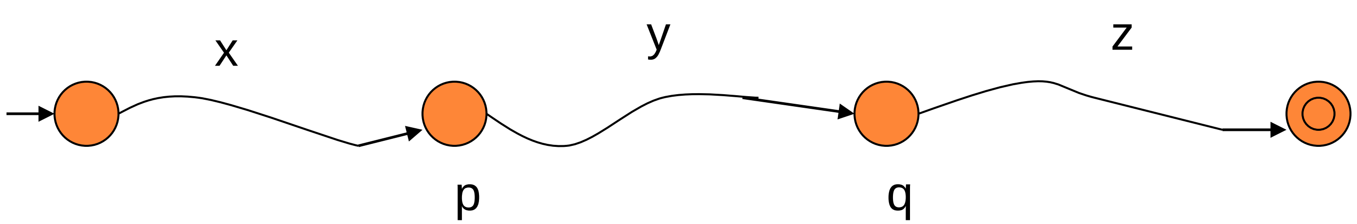
  (2) xuv*wz is a subset of L.

  (for $n \geq 0$, xuv wy in L)

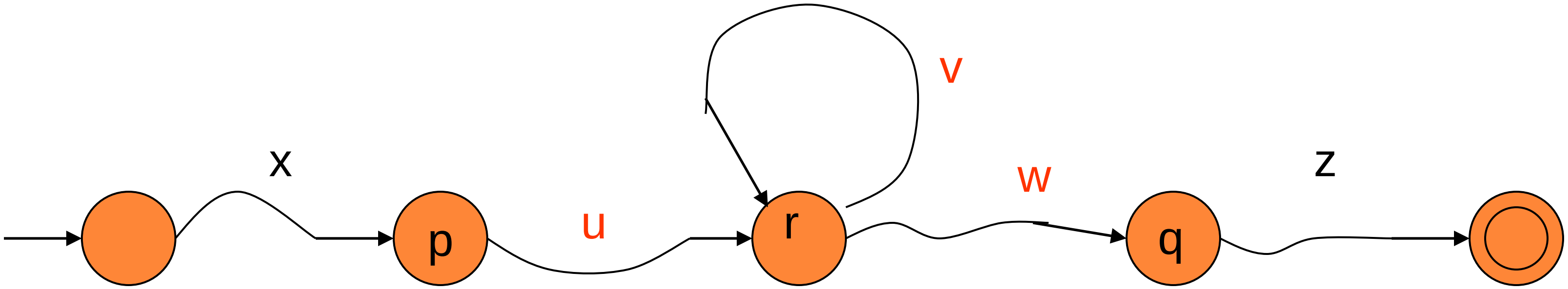
n

Proof

- Since M is DFA, there is a path from initial state to a final state, associated with xyz.



Since $|y| \geq m$, there are at least m+1 nodes between p and q. Hence, there is a state r appearing twice.



v≠ε because M is DFA (without ε-move).

xuwz in L because a path associated with xuwz exists from initial state to a final state.

xuvvwz in L because a path associated with xuvvwz exists from initial state to a final state.

$xuv^n wz$ in L

$L=\{0^i 1^i \mid i \geq 0\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider $0^m 1^m$.

By Pumping Lemma, 0^m = uvw such that $v \neq \varepsilon$ and for $n \geq 0$, $uv^n w1^m$ in L.

$uw1^m$ in L, but uw contains less than m 0's. (-><-)

$L = \{x \text{ in } (0+1)^* \mid \#_1(x) = \#_0(x)\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider $0^m 1^m$.

By Pumping Lemma, $0^m = uvw$ such that $v \neq \varepsilon$ and for $n \geq 0$, $uv^n w1^m$ in L.

$uw1^m$ in L, but uw contains less than m 0's. (-><-)

$L=\{0^i 1^j \mid i \geq j \geq 0\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider $0^m 1^m$.

By Pumping Lemma, 0^m = uvw such that $v \neq \varepsilon$ and for $n \geq 0$, $uv^n w1^m$ in L.

$uw1^m$ in L, but uw contains less than m 0's. (-><-)

$L=\{0^i 1^j \mid i > j \geq 0\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider 00^m1^m.

By Pumping Lemma, $0^m = uvw$ such that $v \neq \varepsilon$ and for $n \geq 0$, $0uv^nw1^m$ in L.

$0uw1^m$ in L, but uw contains less than m 0's. (-><-)

$L=\{a^i b^j c^k \mid i + j = k, i \geq 0, j \geq 0, k \geq 0\}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L= L(M) for some DFA M.

Suppose M has m states. Consider $b^m c^m$.

By Pumping Lemma, b^m = uvw such that $v \neq \varepsilon$ and for $n \geq 0$, $uv^n wc^m$ in L.

uwc^m in L, but uw contains less than m b's. (-><-)

$L = \{ 0^{i^2} \mid i \geq 0 \}$ is not regular.

Proof. For contradiction, suppose L is regular. So, L=L(M) for some DFA M.

Suppose M has m states. Consider 0^{m^2}.

By Pumping Lemma, $0^{m^2} = uvw$ such that $v \neq \varepsilon$ and for $n \geq 0$, uv^nw in L.

Set $a=|v|$ and $b=|uw|$. Then $a > 0$ and for $n \geq 0$, $an+b$ is a square.

Specially, when n=0, b is a square. Set b = cc.

When $n = a+2c$, $an+cc = (a+c)^2$.

Now, consider $n=a+2c+1$.

Note that $an+b = (a+c)^2 + a$.

But, $(a+c+1)^2 = (a+c)^2 + 2(a+c) + 1 > (a+c)^2 + a$.

Hence, $(a+c)^2 + a$ cannot be a square. (-><-)

Puzzle

- $A = \{wx \mid w \in (0+1)^*, x \in (0+1)^+ \text{ and } x = x^R\}$ is regular because it is equal to $(0+1)^*$. In fact, $0 = 0^R$ and $1 = 1^R$.
- $B = \{wx \mid w \in (0+1)^*, x \in (0+1)^+ \text{ and } x = x^R \text{ with odd } |x|\}$ is regular for the same reason.
- $C = \{wx \mid w \in (0+1)^*, x \in (0+1)^+ \text{ and } x = x^R \text{ with even } |x|\}$ is not regular because in this case, $x = yy^R$ so that $C = \{wyy^R \mid w \in (0+1)^*, y \in (0+1)^+\}$.
- Now, $C = A - B$. So, we obtain an example that a regular language subtracts another language and the result is not regular. What's wrong?